

SALESIAN COLLEGE



MATHEMATICS DEPARTMENT

HIT THE GROUND RUNNING NOTES AND EXAMPLES



Expanding brackets and simplifying expressions

A LEVEL LINKS

Scheme of work: 1a. Algebraic expressions – basic algebraic manipulation, indices and surds

Key points

- When you expand one set of brackets you must multiply everything inside the bracket by what is outside.
- When you expand two linear expressions, each with two terms of the form $ax + b$, where $a \neq 0$ and $b \neq 0$, you create four terms. Two of these can usually be simplified by collecting like terms.

Examples

Example 1 Expand $4(3x - 2)$

$$4(3x - 2) = 12x - 8$$

Multiply everything inside the bracket by the 4 outside the bracket

Example 2 Expand and simplify $3(x + 5) - 4(2x + 3)$

$$\begin{aligned} 3(x + 5) - 4(2x + 3) \\ = 3x + 15 - 8x - 12 \\ = 3 - 5x \end{aligned}$$

- 1 Expand each set of brackets separately by multiplying $(x + 5)$ by 3 and $(2x + 3)$ by -4
- 2 Simplify by collecting like terms:
 $3x - 8x = -5x$ and $15 - 12 = 3$

Example 3 Expand and simplify $(x + 3)(x + 2)$

$$\begin{aligned} (x + 3)(x + 2) \\ = x(x + 2) + 3(x + 2) \\ = x^2 + 2x + 3x + 6 \\ = x^2 + 5x + 6 \end{aligned}$$

- 1 Expand the brackets by multiplying $(x + 2)$ by x and $(x + 2)$ by 3
- 2 Simplify by collecting like terms:
 $2x + 3x = 5x$

Example 4 Expand and simplify $(x - 5)(2x + 3)$

$$\begin{aligned} (x - 5)(2x + 3) \\ = x(2x + 3) - 5(2x + 3) \\ = 2x^2 + 3x - 10x - 15 \\ = 2x^2 - 7x - 15 \end{aligned}$$

- 1 Expand the brackets by multiplying $(2x + 3)$ by x and $(2x + 3)$ by -5
- 2 Simplify by collecting like terms:
 $3x - 10x = -7x$



Surds and rationalising the denominator

A LEVEL LINKS

Scheme of work: 1a. Algebraic expressions – basic algebraic manipulation, indices and surds

Key points

- A surd is the square root of a number that is not a square number, for example $\sqrt{2}, \sqrt{3}, \sqrt{5}$, etc.
- Surds can be used to give the exact value for an answer.
- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
- To rationalise the denominator means to remove the surd from the denominator of a fraction.
- To rationalise $\frac{a}{\sqrt{b}}$ you multiply the numerator and denominator by the surd \sqrt{b}
- To rationalise $\frac{a}{b + \sqrt{c}}$ you multiply the numerator and denominator by $b - \sqrt{c}$

Examples

Example 1 Simplify $\sqrt{50}$

$\begin{aligned}\sqrt{50} &= \sqrt{25 \times 2} \\ &= \sqrt{25} \times \sqrt{2} \\ &= 5 \times \sqrt{2} \\ &= 5\sqrt{2}\end{aligned}$	<ol style="list-style-type: none"> Choose two numbers that are factors of 50. One of the factors must be a square number Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ Use $\sqrt{25} = 5$
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Example 2 Simplify $\sqrt{147} - 2\sqrt{12}$

$\begin{aligned}\sqrt{147} - 2\sqrt{12} \\ &= \sqrt{49 \times 3} - 2\sqrt{4 \times 3} \\ &= \sqrt{49} \times \sqrt{3} - 2\sqrt{4} \times \sqrt{3} \\ &= 7 \times \sqrt{3} - 2 \times 2 \times \sqrt{3} \\ &= 7\sqrt{3} - 4\sqrt{3} \\ &= 3\sqrt{3}\end{aligned}$	<ol style="list-style-type: none"> Simplify $\sqrt{147}$ and $2\sqrt{12}$. Choose two numbers that are factors of 147 and two numbers that are factors of 12. One of each pair of factors must be a square number Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ Use $\sqrt{49} = 7$ and $\sqrt{4} = 2$ Collect like terms
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Example 3 Simplify $(\sqrt{7} + \sqrt{2})(\sqrt{7} - \sqrt{2})$

$ \begin{aligned} &(\sqrt{7} + \sqrt{2})(\sqrt{7} - \sqrt{2}) \\ &= \sqrt{49} - \sqrt{7}\sqrt{2} + \sqrt{2}\sqrt{7} - \sqrt{4} \\ &= 7 - 2 \\ &= 5 \end{aligned} $	<p>1 Expand the brackets. A common mistake here is to write $(\sqrt{7})^2 = 49$</p> <p>2 Collect like terms: $-\sqrt{7}\sqrt{2} + \sqrt{2}\sqrt{7} = -\sqrt{7}\sqrt{2} + \sqrt{7}\sqrt{2} = 0$</p>
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Example 4 Rationalise $\frac{1}{\sqrt{3}}$

$ \begin{aligned} \frac{1}{\sqrt{3}} &= \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{1 \times \sqrt{3}}{\sqrt{9}} \\ &= \frac{\sqrt{3}}{3} \end{aligned} $	<p>1 Multiply the numerator and denominator by $\sqrt{3}$</p> <p>2 Use $\sqrt{9} = 3$</p>
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Example 5 Rationalise and simplify $\frac{\sqrt{2}}{\sqrt{12}}$

$ \begin{aligned} \frac{\sqrt{2}}{\sqrt{12}} &= \frac{\sqrt{2}}{\sqrt{12}} \times \frac{\sqrt{12}}{\sqrt{12}} \\ &= \frac{\sqrt{2} \times \sqrt{4 \times 3}}{12} \\ &= \frac{2\sqrt{2}\sqrt{3}}{12} \\ &= \frac{\sqrt{2}\sqrt{3}}{6} \end{aligned} $	<p>1 Multiply the numerator and denominator by $\sqrt{12}$</p> <p>2 Simplify $\sqrt{12}$ in the numerator. Choose two numbers that are factors of 12. One of the factors must be a square number</p> <p>3 Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$</p> <p>4 Use $\sqrt{4} = 2$</p> <p>5 Simplify the fraction: $\frac{2}{12}$ simplifies to $\frac{1}{6}$</p>
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Rules of indices

A LEVEL LINKS

Scheme of work: 1a. Algebraic expressions – basic algebraic manipulation, indices and surds

Key points

- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $a^0 = 1$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$ i.e. the n th root of a
- $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$
- $a^{-m} = \frac{1}{a^m}$
- The square root of a number produces two solutions, e.g. $\sqrt{16} = \pm 4$.

Examples

Example 1 Evaluate 10^0

$10^0 = 1$	Any value raised to the power of zero is equal to 1
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Example 2 Evaluate $9^{\frac{1}{2}}$

$9^{\frac{1}{2}} = \sqrt{9}$ $= 3$	Use the rule $a^{\frac{1}{n}} = \sqrt[n]{a}$
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Example 3 Evaluate $27^{\frac{2}{3}}$

$27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2$ $= 3^2$ $= 9$	<p>1 Use the rule $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$</p> <p>2 Use $\sqrt[3]{27} = 3$</p>
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Example 4 Evaluate 4^{-2}

$4^{-2} = \frac{1}{4^2}$ $= \frac{1}{16}$	<p>1 Use the rule $a^{-m} = \frac{1}{a^m}$</p> <p>2 Use $4^2 = 16$</p>
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Example 5 Simplify $\frac{6x^5}{2x^2}$

$\frac{6x^5}{2x^2} = 3x^3$	<p>$6 \div 2 = 3$ and use the rule $\frac{a^m}{a^n} = a^{m-n}$ to give $\frac{x^5}{x^2} = x^{5-2} = x^3$</p>
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Example 6 Simplify $\frac{x^3 \times x^5}{x^4}$

$\frac{x^3 \times x^5}{x^4} = \frac{x^{3+5}}{x^4} = \frac{x^8}{x^4}$ $= x^{8-4} = x^4$	<p>1 Use the rule $a^m \times a^n = a^{m+n}$</p> <p>2 Use the rule $\frac{a^m}{a^n} = a^{m-n}$</p>
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Example 7 Write $\frac{1}{3x}$ as a single power of x

$\frac{1}{3x} = \frac{1}{3} x^{-1}$	<p>Use the rule $\frac{1}{a^m} = a^{-m}$, note that the fraction $\frac{1}{3}$ remains unchanged</p>
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Example 8 Write $\frac{4}{\sqrt{x}}$ as a single power of x

$\frac{4}{\sqrt{x}} = \frac{4}{x^{\frac{1}{2}}}$ $= 4x^{-\frac{1}{2}}$	<p>1 Use the rule $a^{\frac{1}{n}} = \sqrt[n]{a}$</p> <p>2 Use the rule $\frac{1}{a^m} = a^{-m}$</p>
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Factorising expressions

A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- Factorising an expression is the opposite of expanding the brackets.
- A quadratic expression is in the form $ax^2 + bx + c$, where $a \neq 0$.
- To factorise a quadratic equation find two numbers whose sum is b and whose product is ac .
- An expression in the form $x^2 - y^2$ is called the difference of two squares. It factorises to $(x - y)(x + y)$.

Examples

Example 1 Factorise $15x^2y^3 + 9x^4y$

$$15x^2y^3 + 9x^4y = 3x^2y(5y^2 + 3x^2)$$

The highest common factor is $3x^2y$.

So take $3x^2y$ outside the brackets and then divide each term by $3x^2y$ to find the terms in the brackets

Example 2 Factorise $4x^2 - 25y^2$

$$4x^2 - 25y^2 = (2x + 5y)(2x - 5y)$$

This is the difference of two squares as the two terms can be written as $(2x)^2$ and $(5y)^2$

Example 3 Factorise $x^2 + 3x - 10$

$$b = 3, ac = -10$$

$$\text{So } x^2 + 3x - 10 = x^2 + 5x - 2x - 10$$

$$= x(x + 5) - 2(x + 5)$$

$$= (x + 5)(x - 2)$$

- 1 Work out the two factors of $ac = -10$ which add to give $b = 3$ (5 and -2)
- 2 Rewrite the b term ($3x$) using these two factors
- 3 Factorise the first two terms and the last two terms
- 4 $(x + 5)$ is a factor of both terms



Example 4 Factorise $6x^2 - 11x - 10$

$b = -11, ac = -60$ So $6x^2 - 11x - 10 = 6x^2 - 15x + 4x - 10$ $= 3x(2x - 5) + 2(2x - 5)$ $= (2x - 5)(3x + 2)$	<ol style="list-style-type: none"> 1 Work out the two factors of $ac = -60$ which add to give $b = -11$ (-15 and 4) 2 Rewrite the b term ($-11x$) using these two factors 3 Factorise the first two terms and the last two terms 4 $(2x - 5)$ is a factor of both terms
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Example 5 Simplify $\frac{x^2 - 4x - 21}{2x^2 + 9x + 9}$

$\frac{x^2 - 4x - 21}{2x^2 + 9x + 9}$ For the numerator: $b = -4, ac = -21$ So $x^2 - 4x - 21 = x^2 - 7x + 3x - 21$ $= x(x - 7) + 3(x - 7)$ $= (x - 7)(x + 3)$ For the denominator: $b = 9, ac = 18$ So $2x^2 + 9x + 9 = 2x^2 + 6x + 3x + 9$ $= 2x(x + 3) + 3(x + 3)$ $= (x + 3)(2x + 3)$ So $\frac{x^2 - 4x - 21}{2x^2 + 9x + 9} = \frac{(x - 7)(x + 3)}{(x + 3)(2x + 3)}$ $= \frac{x - 7}{2x + 3}$	<ol style="list-style-type: none"> 1 Factorise the numerator and the denominator 2 Work out the two factors of $ac = -21$ which add to give $b = -4$ (-7 and 3) 3 Rewrite the b term ($-4x$) using these two factors 4 Factorise the first two terms and the last two terms 5 $(x - 7)$ is a factor of both terms 6 Work out the two factors of $ac = 18$ which add to give $b = 9$ (6 and 3) 7 Rewrite the b term ($9x$) using these two factors 8 Factorise the first two terms and the last two terms 9 $(x + 3)$ is a factor of both terms 10 $(x + 3)$ is a factor of both the numerator and denominator so cancels out as a value divided by itself is 1
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Completing the square

A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- Completing the square for a quadratic rearranges $ax^2 + bx + c$ into the form $p(x + q)^2 + r$
- If $a \neq 1$, then factorise using a as a common factor.

Examples

Example 1 Complete the square for the quadratic expression $x^2 + 6x - 2$

$x^2 + 6x - 2$ $= (x + 3)^2 - 9 - 2$ $= (x + 3)^2 - 11$	<p>1 Write $x^2 + bx + c$ in the form $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$</p> <p>2 Simplify</p>
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Example 2 Write $2x^2 - 5x + 1$ in the form $p(x + q)^2 + r$

$2x^2 - 5x + 1$ $= 2\left(x^2 - \frac{5}{2}x\right) + 1$ $= 2\left[\left(x - \frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right] + 1$ $= 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8} + 1$ $= 2\left(x - \frac{5}{4}\right)^2 - \frac{17}{8}$	<p>1 Before completing the square write $ax^2 + bx + c$ in the form $a\left(x^2 + \frac{b}{a}x\right) + c$</p> <p>2 Now complete the square by writing $x^2 - \frac{5}{2}x$ in the form $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$</p> <p>3 Expand the square brackets – don't forget to multiply $\left(\frac{5}{4}\right)^2$ by the factor of 2</p> <p>4 Simplify</p>
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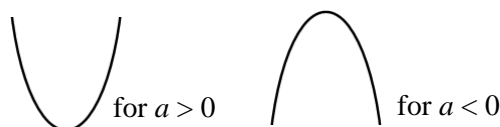
Sketching quadratic graphs

A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- The graph of the quadratic function $y = ax^2 + bx + c$, where $a \neq 0$, is a curve called a parabola.
- Parabolas have a line of symmetry and a shape as shown.
- To sketch the graph of a function, find the points where the graph intersects the axes.
- To find where the curve intersects the y-axis substitute $x = 0$ into the function.
- To find where the curve intersects the x-axis substitute $y = 0$ into the function.
- At the turning points of a graph the gradient of the curve is 0 and any tangents to the curve at these points are horizontal.
- To find the coordinates of the maximum or minimum point (turning points) of a quadratic curve (parabola) you can use the completed square form of the function.



Examples

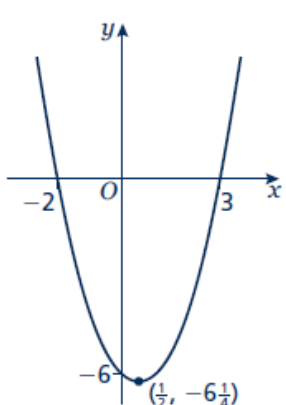
Example 1 Sketch the graph of $y = x^2$.

	<p>The graph of $y = x^2$ is a parabola.</p> <p>When $x = 0$, $y = 0$.</p> <p>$a = 1$ which is greater than zero, so the graph has the shape:</p>
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Example 2 Sketch the graph of $y = x^2 - x - 6$.

<p>When $x = 0$, $y = 0^2 - 0 - 6 = -6$ So the graph intersects the y-axis at $(0, -6)$ When $y = 0$, $x^2 - x - 6 = 0$ $(x + 2)(x - 3) = 0$ $x = -2$ or $x = 3$</p> <p>So, the graph intersects the x-axis at $(-2, 0)$ and $(3, 0)$</p>	<ol style="list-style-type: none"> Find where the graph intersects the y-axis by substituting $x = 0$. Find where the graph intersects the x-axis by substituting $y = 0$. Solve the equation by factorising. Solve $(x + 2) = 0$ and $(x - 3) = 0$. $a = 1$ which is greater than zero, so the graph has the shape: <p><i>(continued on next page)</i></p>
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$x^2 - x - 6 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - 6$ $= \left(x - \frac{1}{2}\right)^2 - \frac{25}{4}$ <p>When $\left(x - \frac{1}{2}\right)^2 = 0$, $x = \frac{1}{2}$ and</p> <p>$y = -\frac{25}{4}$, so the turning point is at the point $\left(\frac{1}{2}, -\frac{25}{4}\right)$</p> 	<p>6 To find the turning point, complete the square.</p> <p>7 The turning point is the minimum value for this expression and occurs when the term in the bracket is equal to zero.</p>
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Solving linear simultaneous equations using the substitution method

A LEVEL LINKS

Scheme of work: 1c. Equations – quadratic/linear simultaneous

Textbook: Pure Year 1, 3.1 Linear simultaneous equations

Key points

- The substitution method is the method most commonly used for A level. This is because it is the method used to solve linear and quadratic simultaneous equations.

Examples

Example 4 Solve the simultaneous equations $y = 2x + 1$ and $5x + 3y = 14$

$5x + 3(2x + 1) = 14$ $5x + 6x + 3 = 14$ $11x + 3 = 14$ $11x = 11$ $\text{So } x = 1$ $\text{Using } y = 2x + 1$ $y = 2 \times 1 + 1$ $\text{So } y = 3$ Check: $\text{equation 1: } 3 = 2 \times 1 + 1 \quad \text{YES}$ $\text{equation 2: } 5 \times 1 + 3 \times 3 = 14 \quad \text{YES}$	<ol style="list-style-type: none"> 1 Substitute $2x + 1$ for y into the second equation. 2 Expand the brackets and simplify. 3 Work out the value of x. 4 To find the value of y, substitute $x = 1$ into one of the original equations. 5 Substitute the values of x and y into both equations to check your answers.
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Example 5 Solve $2x - y = 16$ and $4x + 3y = -3$ simultaneously.

$y = 2x - 16$ $4x + 3(2x - 16) = -3$ $4x + 6x - 48 = -3$ $10x - 48 = -3$ $10x = 45$ $\text{So } x = 4\frac{1}{2}$ $\text{Using } y = 2x - 16$ $y = 2 \times 4\frac{1}{2} - 16$ $\text{So } y = -7$ Check: $\text{equation 1: } 2 \times 4\frac{1}{2} - (-7) = 16 \quad \text{YES}$ $\text{equation 2: } 4 \times 4\frac{1}{2} + 3 \times (-7) = -3 \quad \text{YES}$	<ol style="list-style-type: none"> 1 Rearrange the first equation. 2 Substitute $2x - 16$ for y into the second equation. 3 Expand the brackets and simplify. 4 Work out the value of x. 5 To find the value of y, substitute $x = 4\frac{1}{2}$ into one of the original equations. 6 Substitute the values of x and y into both equations to check your answers.
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Solving linear and quadratic simultaneous equations

A LEVEL LINKS

Scheme of work: 1c. Equations – quadratic/linear simultaneous

Key points

- Make one of the unknowns the subject of the linear equation (rearranging where necessary).
- Use the linear equation to substitute into the quadratic equation.
- There are usually two pairs of solutions.

Examples

Example 1 Solve the simultaneous equations $y = x + 1$ and $x^2 + y^2 = 13$

$x^2 + (x + 1)^2 = 13$ $x^2 + x^2 + x + x + 1 = 13$ $2x^2 + 2x + 1 = 13$ $2x^2 + 2x - 12 = 0$ $(2x - 4)(x + 3) = 0$ So $x = 2$ or $x = -3$ Using $y = x + 1$ When $x = 2$, $y = 2 + 1 = 3$ When $x = -3$, $y = -3 + 1 = -2$ So the solutions are $x = 2, y = 3$ and $x = -3, y = -2$ Check: equation 1: $3 = 2 + 1$ YES and $-2 = -3 + 1$ YES equation 2: $2^2 + 3^2 = 13$ YES and $(-3)^2 + (-2)^2 = 13$ YES	<ol style="list-style-type: none"> 1 Substitute $x + 1$ for y into the second equation. 2 Expand the brackets and simplify. 3 Factorise the quadratic equation. 4 Work out the values of x. 5 To find the value of y, substitute both values of x into one of the original equations. 6 Substitute both pairs of values of x and y into both equations to check your answers.
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Example 2 Solve $2x + 3y = 5$ and $2y^2 + xy = 12$ simultaneously.

$x = \frac{5-3y}{2}$ $2y^2 + \left(\frac{5-3y}{2}\right)y = 12$ $2y^2 + \frac{5y-3y^2}{2} = 12$ $4y^2 + 5y - 3y^2 = 24$ $y^2 + 5y - 24 = 0$ $(y+8)(y-3) = 0$ <p>So $y = -8$ or $y = 3$</p> <p>Using $2x + 3y = 5$ When $y = -8$, $2x + 3 \times (-8) = 5$, $x = 14.5$ When $y = 3$, $2x + 3 \times 3 = 5$, $x = -2$</p> <p>So the solutions are $x = 14.5$, $y = -8$ and $x = -2$, $y = 3$</p> <p>Check: equation 1: $2 \times 14.5 + 3 \times (-8) = 5$ YES and $2 \times (-2) + 3 \times 3 = 5$ YES equation 2: $2 \times (-8)^2 + 14.5 \times (-8) = 12$ YES and $2 \times (3)^2 + (-2) \times 3 = 12$ YES</p>	<ol style="list-style-type: none"> 1 Rearrange the first equation. 2 Substitute $\frac{5-3y}{2}$ for x into the second equation. Notice how it is easier to substitute for x than for y. 3 Expand the brackets and simplify. 4 Factorise the quadratic equation. 5 Work out the values of y. 6 To find the value of x, substitute both values of y into one of the original equations. 7 Substitute both pairs of values of x and y into both equations to check your answers.
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